

Talbot Oscillations and Periodic Focusing in a One-Dimensional Condensate

A.G. Rojo, J.L. Cohen, and P.R. Berman

Department of Physics, University of Michigan, Ann Arbor, MI 48109-1120

(November 30, 1998)

An exact theory for the density of a one-dimensional Bose-Einstein condensate with hard core particle interactions is developed in second quantization and applied to the scattering of the condensate by a spatially periodic impulse potential. The boson problem is mapped onto a system of free fermions obeying the Pauli exclusion principle to facilitate the calculation. The density exhibits a spatial focusing of the probability density as well as a periodic self-imaging in time, or Talbot effect. Furthermore, the transition from single particle to many body effects can be measured by observing the decay of the modulated condensate density pattern in time. The connection of these results to classical and atom optical phase gratings is made explicit.

I. INTRODUCTION

The recent interest in neutral atom Bose-Einstein condensates [1,2] and in periodic effects [3–5] in atom optics has stimulated research in those fields. Experiments are at the point now where the condensate can be manipulated [6] and used for precision measurement and atom optical studies. From a theoretical perspective much can be learned by applying the methods developed over the past four decades in condensed matter research to these fields. Similarly, new effects from a condensed matter viewpoint can be introduced by adopting the knowledge gained over the years in mathematical physics, optics, and atomic physics. With this motivation we detail the theory of a one-dimensional condensate with hard-core particle-particle interactions subjected to a spatially periodic, impulsive interaction. The subsequent evolution of the system probes the onset of many-body effects represented by the interactions.

The results are summarized as follows. The condensate of interacting bosons can be mapped onto a system of free fermions obeying the Pauli exclusion principle. The boson-boson interaction in one-dimension acts like the Fermi repulsion to create a parabolic band of occupied states. If we are interested in observables, such as the density, which probe the diagonal density matrix elements of the system, the difference between the fermion and boson commutation relations is irrelevant to our results.

Using this fermion model, we show that an exact expression for the density after the periodic interaction can be derived for the case of a pulse with duration τ such that $\max[\epsilon_{k_F}\tau, \epsilon_q\tau] \ll \hbar$, where the Fermi and recoil energies, ϵ_{k_F} and ϵ_q , respectively, are defined below. This density describes both the one-dimensional interacting bosons and the noninteracting (degenerate) Fermi gas. Since the density becomes spatially modulated after the interaction, regions of high density, or focal spots, can arise when the spatial harmonics at a certain time superpose *in phase* at certain points in space and *out of*

phase in others. In addition, the spatially modulated density exhibits a Talbot effect, resulting from the periodic self-imaging or wave packet revival of each fermion's time-dependent, single particle wave function. Simultaneously, the inhomogeneous dephasing due to the initial Fermi distribution of momenta causes a decay in time of the spatial modulation. Since the Talbot effect is a coherent, single particle phenomenon and the distribution of initial momenta represents the interactions in the system, in some sense the decay of the modulated density in the condensate measures the crossover from single particle to many-body physics in an interacting system.

II. THE MODEL

We consider N bosons with hard core interactions in a one-dimensional “box” of length L with periodic boundary conditions. The Hamiltonian H for this system corresponds to particles with mass M and radius $a \rightarrow 0$ interacting via effective point or δ -function interactions in the limit of infinite interaction strength:

$$H = -\frac{\hbar^2}{2M} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{\eta}{2} \sum_{i \neq j} \delta(x_i - x_j), \quad \text{with } \eta \rightarrow \infty. \quad (1)$$

The effect of the interactions is to introduce nodes in the many-particle eigenstate wave functions whenever two coordinates coincide: the bosons are impenetrable. The problem of impenetrable bosons in one dimension has a very close relationship to the problem of *free* fermions. The energy eigenfunctions differ only by a multiplicative function with value ± 1 which symmetrizes the anti-symmetric fermion eigenstates to form boson states, and the corresponding eigenvalues are identical [7,8]. The eigenstates can be labeled by a set of wave vectors $\{k_i\}$ with $i = 1, \dots, N$, $k_i = \frac{2\pi}{L}n_i$, and n_i integers satisfying $n_i \neq n_j$. We take N odd. The N even case can be treated similarly [8].

For free fermions the ground state wave function is the following Slater determinant of plane waves,

$$\Psi_0^{(F)}(x_1, x_2, \dots, x_N) = \frac{1}{\sqrt{N!L^N}} \det_{(1-N)/2 \leq n_j \leq (N-1)/2, 1 \leq m \leq N} e^{2\pi i n_j x_m / L}. \quad (2)$$

The fermionic wave function corresponds to occupying N one-particle states with the lowest energy satisfying the Pauli exclusion principle. The result is a “Fermi sea” of occupied states in the interval $k \in [-k_F, k_F]$ for

$$k_F/\pi \equiv (N-1)/L = \rho_0 \equiv 1/d, \quad (3)$$

where ρ_0 is the mean density and d is the interparticle distance. The excited states are constructed by emptying some states of the Fermi sea and occupying states with

$|k| > k_F$. The corresponding bosonic wave function can be written simply as

$$\Psi_0^{(B)}(x_1, x_2, \dots, x_N) = \frac{1}{\sqrt{N!L^N}} \left| \det_{(1-N)/2 \leq n_j \leq (N-1)/2, 1 \leq m \leq N} e^{2\pi i n_j x_m / L} \right| \quad (4a)$$

$$= \frac{1}{\sqrt{N!L^N}} \prod_{1 \leq n < m \leq N} \left| e^{2\pi i x_n / L} - e^{2\pi i x_m / L} \right|. \quad (4b)$$

Since the bosonic wave function (for all the states of the spectrum) is even under permutation of any pair of coordinates x_i with x_j , an equivalent description of the ground state wave function is

$$\Psi_0^{(B)}(x_1 < x_2 < \dots < x_N) = \frac{1}{\sqrt{L^N}} \det_{(1-N)/2 \leq n_j \leq (N-1)/2, 1 \leq m \leq N} e^{2\pi i n_j x_m / L}. \quad (5)$$

In second quantization, the Hamiltonians for fermions and impenetrable (or hard-core) bosons are very similar:

$$H_F = \sum_k \epsilon_k c_k^\dagger c_k, \quad (6a)$$

$$H_B = \sum_k \epsilon_k a_k^\dagger a_k, \quad (6b)$$

where

$$c_k = \frac{1}{\sqrt{L}} \int_0^L dx e^{-ikx} c_x, \quad a_k = \frac{1}{\sqrt{L}} \int_0^L dx e^{-ikx} a_x, \quad (7)$$

$\epsilon_k = k^2/2$ is the kinetic energy with $M, \hbar \rightarrow 1$, and the field operators c_x and a_x destroy a fermion and a boson, respectively, at site x . Despite their similarity, there is a basic difference in the commutation relation obeyed by such operators, $\{c_x, c_{x'}^\dagger\} = \delta(x - x')$ and $[a_x, a_{x'}^\dagger] = \delta(x - x')$, supplemented by the hard-core condition $(a_x^\dagger)^2 = 0$, where $[\cdot, \cdot]$ is the commutator and $\{\cdot, \cdot\}$ is the anti-commutator. In second quantization the hard-core interaction has been absorbed into the definition of the field operators which must be used to construct any state. This is a key point. As a result of the condition $(a_x^\dagger)^2 = 0$, the boson ground state cannot be easily constructed from the vacuum using the creation and annihilation operators, a_k^\dagger and a_k . Furthermore, we do not know how to calculate the state rotation caused by interaction operators (*i.e.*, a pulse) on arbitrary boson states. However, the fermion energy eigenstates are well known, and the ability to determine the action of operators on any fermion state allows us to perform tractable calculations.

The state vectors written in the (real) Fock space take the same form for fermions and bosons. For example, take a state consisting of three particles, $|x, x', x''\rangle$, with $x < x' < x''$. The non-trivial difference appears in the

evaluation of off-diagonal matrix elements. For the previous state,

$$|x, x', x''\rangle_F = c_x^\dagger c_{x'}^\dagger c_{x''}^\dagger |\emptyset\rangle, \quad (8a)$$

$$|x, x', x''\rangle_B = a_x^\dagger a_{x'}^\dagger a_{x''}^\dagger |\emptyset\rangle, \quad (8b)$$

the following off-diagonal matrix elements illustrate the difference in question ($x_0 < x$),

$$\langle x_0, x, x'' | c_0^\dagger c_{x'} | x, x', x'' \rangle_F = -1, \quad (8c)$$

$$\langle x_0, x, x'' | a_0^\dagger a_{x'} | x, x', x'' \rangle_B = +1. \quad (8d)$$

This difference arises from the commutation relations. As a direct result, the commutation relations allow for the possibility of having more than one interacting boson in the $k = 0$ state as long as two bosons are not at the same position, whereas for free fermions the exclusion principle limits the occupation probability of the $k = 0$ state to one or zero.

Information about the condensate in this one-dimensional model is contained in the (single particle) density matrix,

$$\rho(x, x', t) = \langle \Psi(t) | a_x^\dagger a_{x'} | \Psi(t) \rangle. \quad (9)$$

For the ground state, the density matrix,

$$\rho(x, x') = \langle \Psi_0 | a_x^\dagger a_{x'} | \Psi_0 \rangle, \quad (10)$$

was analyzed by Lenard [8]. The condensate wave function $\Psi_c(x)$ is defined as the eigenstate of $\rho(x, x')$ with the highest eigenvalue N_c : $\int_0^L dx' \rho(x, x') \Psi_c(x') = N_c \Psi_c(x)$.

According to the Penrose and Onsager criterion [9], a true Bose–Einstein condensate corresponds to $N_c \sim \mathcal{O}(N)$. For the present one-dimensional case with hard core interactions, Lenard showed that $N_c \sim \mathcal{O}(\sqrt{N})$ [8]. This implies that in the thermodynamic limit the particle *density* vanishes in the condensate. However, for a finite system there can still be a large *number* of particles in the condensate. This vanishing density is connected to the fact that there is no Bose–Einstein condensation in one–dimension for a free Bose gas at finite temperature. Even though we are considering here the limit of zero temperature, the quantum fluctuations generated by the hard core interactions are enough to destroy the condensate in the limit of infinite particle number. Given the above proviso, our model is relevant for the understanding of interactions since an infinite, non–intensive number of particles remains in the lowest lying state.

For fermions the ground state wave function corresponding to the Fermi sea, $\Psi_0^{(F)}$ [Eq. (2)], can be written in second quantized form as the state vector

$$|F\rangle = \prod_{|k| \leq k_F} c_k^\dagger |\emptyset\rangle, \quad (11)$$

and the ground state density matrix, Eq. (10) with $a_x \rightarrow c_x$, can be evaluated easily as

$$\rho_F(x, x') = \frac{1}{L} \sum_k f_k e^{ik(x'-x)}. \quad (12)$$

The Fermi factor f_k satisfies $f_k = 1$ for $|k| \leq k_F$ and zero otherwise. From Eq. (12) the eigenstates of the fermionic density matrix are plane waves e^{iqx} with eigenvalues f_q .

The ground state energy, given by $\langle F | \sum_k \epsilon_k c_k^\dagger c_k | F \rangle$, is $E_0^{(F)} = E_0^{(B)} \simeq N(\hbar k_F)^2 / (6M)$.

We are interested in the time dependence of the boson state vector $|\Psi(t)\rangle$ and density, $\rho(x, t) \equiv \rho(x, x, t)$, after a pulse is applied with the form of an impulsive, periodic phase grating,

$$H'(t) = -2\lambda\delta(t) \sum_{i=1}^N \cos qx_i. \quad (13)$$

One can prove [see Appendix A] that the action of this pulse on either the fermion or boson ground state leads to the same time-dependent, spatially-modulated density at zero temperature given the above connection between fermion and boson eigenstates and eigenenergies. The crucial feature in the proof is that both the pulse operator and the diagonal elements of the density matrix depend only on the boson coordinates (the position x and the diagonal operator $a_x^\dagger a_x$ in second quantization). These operators preserve the symmetrization of the fermion states to form boson states. Therefore, while the occupation probability of k -states is distinct for the interacting bosons and free fermions, the density of the Fermi system mirrors the collective behavior of the many-body (correlated) boson wave function for particles separated on average by a distance d . Furthermore, the fact that the fermions act as free particles enables us to calculate the results for the interacting Bose gas exactly.

From now on we will consider the pulse acting on a system of fermions, the complete Hamiltonian in second quantization being

$$H = \sum_k \epsilon_k c_k^\dagger c_k - \lambda\delta(t) (\rho_q + \rho_q^\dagger) \equiv H_0 - \lambda\delta(t) (\rho_q + \rho_q^\dagger), \quad (14)$$

where

$$\rho_q^\dagger = \sum_k c_k^\dagger c_{k+q} \text{ and } \rho_q = \rho_{-q}^\dagger. \quad (15)$$

Note that we are omitting the subscript F for fermions and that the pulse Hamiltonian in Eq. (14) is the second quantized version of $H'(t)$. The following commutation relations and identity, derived from $\{c_k, c_{k'}^\dagger\} = \delta_{k,k'}$ and $\{c_k, c_{k'}\}_{k \neq k'} = 0$ as well as Eqs. (7) and (15), will prove useful below:

$$[c_k, \rho_q] = c_{k-q}, \quad (16a)$$

$$[c_k, \rho_q^\dagger] = c_{k+q}, \quad (16b)$$

$$c_x = \frac{1}{\sqrt{L}} \sum_k c_k e^{ikx}. \quad (16c)$$

III. RESPONSE TO THE PULSE

For times $t < 0$ the wave function is the Fermi sea,

$$|\Psi(t < 0)\rangle = |F\rangle. \quad (17)$$

For $t = 0^+$ we have from the integration of the Schrödinger equation (14) in the impulse approximation,

$$|\Psi(t = 0^+)\rangle = e^{i\lambda(\rho_q + \rho_q^\dagger)}|F\rangle, \quad (18)$$

and for times $t > 0$

$$|\Psi(t)\rangle \equiv e^{-iH_0 t} e^{-i\lambda(\rho_q + \rho_q^\dagger)}|F\rangle. \quad (19)$$

We are interested in the density $\rho(x, t)$ from Eq. (9) for positive times, which is given by

$$\rho(x, t) = \langle \Psi(t) | c_x^\dagger c_x | \Psi(t) \rangle \quad (20a)$$

$$= \frac{1}{L} \sum_{k, k'} \langle \Psi(t) | c_k^\dagger c_{k'} | \Psi(t) \rangle e^{i(k' - k)x} \quad (20b)$$

$$= \frac{1}{L} \sum_{k, k'} \langle F | e^{-i\lambda(\rho_q + \rho_q^\dagger)} c_k^\dagger c_{k'} e^{i\lambda(\rho_q + \rho_q^\dagger)} | F \rangle e^{i[(k' - k)x + (\epsilon_k - \epsilon_{k'})t]}, \quad (20c)$$

where the third line follows from the second using $e^{iH_0 t} c_k e^{-iH_0 t} = c_k e^{-i\epsilon_k t}$. The problem has been reduced to simplifying the operator $e^{-i\lambda(\rho_q + \rho_q^\dagger)} c_k^\dagger c_{k'} e^{i\lambda(\rho_q + \rho_q^\dagger)}$. Each term in its power series expansion couples the Fermi sea to states within and outside of the Fermi sea. The key to this calculation is that we can evaluate this operator explicitly, summing the effect of the pulsed interaction to all orders in λ .

Rewriting $e^{-i\lambda(\rho_q + \rho_q^\dagger)} c_k^\dagger c_{k'} e^{i\lambda(\rho_q + \rho_q^\dagger)}$ as $e^{-i\lambda(\rho_q + \rho_q^\dagger)} c_k^\dagger e^{i\lambda(\rho_q + \rho_q^\dagger)} e^{-i\lambda(\rho_q + \rho_q^\dagger)} c_{k'} e^{i\lambda(\rho_q + \rho_q^\dagger)}$, we define

$$c_k(\lambda) \equiv e^{-i\lambda(\rho_q + \rho_q^\dagger)} c_k e^{i\lambda(\rho_q + \rho_q^\dagger)}, \quad (21)$$

where the q -dependence of $c_k(\lambda)$ is implicit. Taking the derivative with respect to λ and using the commutation relations (16a) and (16b),

$$\begin{aligned} \frac{dc_k(\lambda)}{d\lambda} &= i e^{-i\lambda(\rho_q + \rho_q^\dagger)} [c_k, \rho_q + \rho_q^\dagger] e^{i\lambda(\rho_q + \rho_q^\dagger)} \\ &= i(c_{k-q}(\lambda) + c_{k+q}(\lambda)). \end{aligned} \quad (22)$$

The solution to this equation with the condition $c_k(\lambda = 0) = c_k$ is

$$c_k(\lambda) = \sum_{s=-\infty}^{\infty} i^s J_s(2\lambda) c_{k-sq}, \quad (23)$$

where J_s is a Bessel function of integer order s .

The action of the pulse transforms the single fermion annihilation operator into a coherent superposition of single fermion operators. In fact, from Eq. (14) we can confirm that $c_k(\lambda) e^{-i\epsilon_k t}$ is the time-dependent Heisenberg annihilation operator. Substituting

$$\langle F | c_k^\dagger(\lambda) c_{k'}(\lambda) | F \rangle = \sum_{s, s'} i^{s'-s} J_s(2\lambda) J_{s'}(2\lambda) \delta_{k'-s'q, k-sq} f_{k'-s'q}$$

into Eq. (20c), letting $k - sq \longrightarrow k$ and $k' - s'q \longrightarrow k'$, and summing over k' , we find

$$\rho(x, t) = \frac{1}{L} \sum_k \sum_{s, s'} i^{s'-s} J_s(2\lambda) J_{s'}(2\lambda) e^{i[(s'-s)qx + (\epsilon_{k+sq} - \epsilon_{k+s'q})t]} f_k. \quad (24)$$

From this equation the calculation shows that, as far as the density is concerned, each particle with initial momentum $\hbar k$ acts independently, multiply-scattering the pulsed interaction to form momentum components $\hbar(k + sq)$. The components of the perturbed particle then propagate to time t with their free energies, ϵ_{k+sq} . The total density at t is formed by the interference of each particle's individual momentum components, averaged over the initial momentum distribution f_k . This averaging process determines whether collective effects owing to particle-particle interactions dominate ($q \ll 2k_F$) or whether the coherent interference of each particle with itself dominates ($q \gg 2k_F$). The dephasing caused by interactions results in a decay of the density pattern as a function of time.

Transforming $2\pi \sum_k \longrightarrow L \int dk$, we can perform the integral by noting the Fermi function f_k restricts the limits of integration to the interval, $-k_F \leq k \leq k_F$. Therefore,

$$\frac{1}{L} \sum_k f_k e^{i[(\epsilon_{k+sq} - \epsilon_{k+s'q})t]} = \frac{1}{2\pi} e^{iq^2(s^2 - s'^2)t/2} \int_{-k_F}^{k_F} dk e^{ikq(s-s')t} \quad (25a)$$

$$= \frac{1}{\pi} e^{iq^2(s^2 - s'^2)t/2} \frac{\sin[k_F q(s' - s)t]}{q(s' - s)t}, \quad (25b)$$

and we can make a change of variable in Eqs. (24) and (25b), $s' - s = j$, to give

$$\rho(x, t) = \frac{1}{\pi} \sum_j i^j e^{i[jqx - j^2 q^2 t/2]} \frac{\sin[k_F q j t]}{q j t} \sum_s J_s(2\lambda) J_{s+j}(2\lambda) e^{-isq^2 j t}. \quad (26)$$

Using the sum rule [10]

$$\sum_s J_s(2\lambda) J_{s+j}(2\lambda) e^{-2is\varphi} = i^{-j} J_j(4\lambda \sin \varphi) e^{ij\varphi} \quad (27)$$

for $\varphi = jq^2 t/2$ yields the desired result,

$$\rho(x, t) = \frac{1}{\pi} \sum_{j=-\infty}^{\infty} J_j(4\lambda \sin jq^2 t/2) e^{ipqx} \frac{\sin[k_F q j t]}{q j t} \quad (28)$$

or

$$\rho(x, t) = \rho_0 + 2 \sum_{j=1}^{\infty} J_j(4\lambda \sin jq^2 t/2m) \cos(jqx) I_j(t). \quad (29)$$

The total particle density is spatially modulated as a function of time, and each spatial harmonic decays in time according to the decoherence function,

$$I_j(t) = \frac{1}{\pi} \frac{\sin[k_F q j t]}{q j t} = \frac{1}{\pi} \frac{\sin[\hbar k_F q j t / M]}{\hbar q j t / M}, \quad (30)$$

owing to the uncertainty in the initial k -vector within the Fermi sea. We have reinserted the \hbar and M to stress the time scales of the solution. We now examine two regimes of this exact solution, $q \gg 2k_F$ (where collective effects are negligible) and $q \lesssim 2k_F$ (where collective effects degrade the density modulation).

IV. RESULTS

A. $q \gg 2k_F$

For the $q \gg 2k_F$ case when $\hbar k_F q j t / M \ll 1$ but $\hbar q^2 j t / (2M)$ is arbitrary, we can ignore the decoherence by setting $I_j(t) \rightarrow I_0 = k_F / \pi = \rho_0$, the background density. In this limit the total density is a periodic function of time with period $\tau_T = 2\pi / (\hbar q^2 / (2M))$. This is the Talbot effect from classical and atom optics, where the periodicity of a diffraction grating is imposed on a transmitted wave, and τ_T is the Talbot period [10–12]. In those cases the transverse wave intensity or density in the Fresnel regime of diffraction becomes periodic in space and time in exactly the manner of Eq. (29). The initial, uniform density evolves into a spatially-modulated density and back again.

If $4\lambda \gtrsim 1$, a third time scale, $\tau_f = 2M / (4\lambda \hbar q^2) = \tau_T / (8\pi\lambda)$, is implicit in the expression for the density, Eq. (29). For $t \sim \tau_f$ the particle distribution is focused to an array of focal spots with density peaks of magnitude $> \rho_0$ at the positions $qx_f = 2\pi m$ for all integers m [12,13]. As a result of the Talbot effect, these focuses also reappear at later times $t = n\tau_T / 2 \pm \tau_f$ for integers $n > 0$. The focal positions qx_f are shown in Table 1.

qx_f	n odd	n even
$t = n\tau_T / 2 + \tau_f$	$2\pi m - \pi$	$2\pi m$
$t = n\tau_T / 2 - \tau_f$	$2\pi m$	$2\pi m - \pi$

Table 1. Focal positions for $4\lambda \gtrsim 1$, $q \gg 2k_F$

Hence, the spatially-modulated pulse acts on the many-body wave function as a periodic array of lenses.

In Fig. 1a we plot the normalized particle density, $\rho(x, t) / \rho_0$, versus $|qx| / \pi \leq 1$ on the horizontal axis and $0 \leq t / \tau_T \leq 0.5$ on the vertical for $\lambda = 2.5$ and $k_F \rightarrow 0$. The highest density spots are white while the lowest are black. The density $\rho(x, t) / \rho_0$ at $t / \tau_T = 0, 0.5$ is one. Furthermore, we can see that the density in this time period is symmetric with respect to $t / \tau_T = 0.25$. The white spot near $x = 0$ and $t / \tau_T \sim \tau_f / \tau_T \simeq 0.016$ is the first focus; the white spot near $x = 0$ and $t / \tau_T \sim 1/2 - \tau_f / \tau_T$ is the second. The density for the time period $0.5 \leq t / \tau_T \leq 1$

(not shown) is identical to Fig. 1a, shifted by half of the spatial period. In Fig. 2a the normalized density versus time, $\rho(qx = 2\pi m, t)/\rho_0$, is plotted for $\lambda = 2.5$ and $k_F \rightarrow 0$. This is the slice of Fig. 1a along $x = 0$. The Talbot effect is clear as the amplitude fluctuations repeat with period τ_T .

B. $q \lesssim 2k_F$

This situation corresponds to the limit for which the momentum “kicks” due to the applied pulse are smaller than the momentum uncertainty in the ground state, and one expects the Talbot effect to be washed out for higher harmonics. From $I_j(t)$ we see that the largest harmonic that survives up to the Talbot time is that for which $k_F q j \tau_T \lesssim \pi$, implying $j \sim q/(4k_F)$. Higher harmonics die away before the pattern can be “reconstructed” at τ_T . In quasi-classical language, a particle that absorbs momentum $j\hbar q$ takes a time $\tau_c \sim d/(\hbar j q/M)$ to travel the inter-particle distance $d = \pi/k_F$ and collide with its neighboring particle. If τ_c is much longer than the Talbot time τ_T , the particle acts as a free particle for multiple Talbot times, and the response of that harmonic will be insensitive to the particle-particle interactions. The condition $\tau_c \sim \tau_T$ gives an estimate of the highest harmonic j_{\max}^T surviving up to the Talbot time,

$$j_{\max}^T \sim \frac{q}{4k_F}, \quad (31)$$

which agrees with the consideration $k_F q j \tau_T \lesssim \pi$ above from the exact result.

To continue the analogy with classical and atom optics [10,14] including the decoherence factor $I_j(t)$, the density of Eq. (29) is isomorphic to the behavior of the atomic density when a beam of neutral atoms with beam divergence $\theta = 2\hbar k_F/M_a v_z \ll 1$ passes through a standing wave phase grating with periodicity $2\pi/q$, where M_a is the atomic mass, $v_z = z/t$ is the longitudinal beam speed, and $2\hbar k_F/M_a$ plays the role of the transverse velocity spread u_x . In that case, the probability distribution of transverse atomic velocities is taken as $P(v_x) = 1/u_x$ for $v_x \in [-u_x/2, u_x/2]$, just as the occupation of k -states in the Fermi sea is $P(k) = 1/(2k_F)$ for $k \in [-k_F, k_F]$ in one-dimension. A similar correspondence holds for the divergence angle, $\theta = 2k_F/k_z$, of a light beam in the paraxial approximation passing through a periodic optical phase grating. As a result, $I_j(t)$ corresponds to the Doppler dephasing caused by a uniform, inhomogeneous distribution of particles with initial wave vectors between $\pm k_F$ with respect to the momentum kick wave vectors $j q$. The sinusoidal behavior of $I_j(t)$ arises from the uniformity of the Fermi distribution over a finite interval of k -states. Other inhomogeneous distribution functions, such as a thermal distribution of particle velocities in the atom beam case, may lead to a smooth decay in time.

Again, if $4\lambda \gtrsim 1$, we can have the situation where $\tau_f \lesssim \tau_c \ll \tau_T$, implying the particle distribution comes

to its first focus before the spatial modulation washes out. Actually, the stricter condition, $\tau_f(2j\hbar k_F/M) \lesssim (4\lambda)^{-3/4} 4\pi/q$, is required. This condition estimates the highest harmonic contributing to a focus for which the spot size for $k_F \rightarrow 0$, $w \sim 4\pi(4\lambda)^{-3/4}/q$, is not broadened by the Fermi distribution of momenta [14]. As a result, we have

$$j_{\max}^f \sim 2\pi(4\lambda)^{1/4} \frac{q}{2k_F}. \quad (32)$$

Even if $q < 2k_F$, j_{\max}^f can be significantly greater than one for $2\pi(4\lambda)^{1/4} \gg 2k_F/q$.

In Fig. 1b we plot $\rho(x, t)/\rho_0$ versus $|qx|/\pi \leq 1$ and t/τ_T for $\lambda = 2.5$ again but now $k_F/q = 0.1$. The pure Talbot effect is washed out as the modulation harmonics of the density clearly damp in time, and the symmetry with respect to $t/\tau_T = 0.25$ is broken. The first focus is still apparent while the second is vague. This is a regime where the initial focusing occurs for all significant j (i.e., $j_{\max}^f \sim 28$ and $J_{28}(4\lambda) \ll 1$). In Fig. 2b the normalized density, $\rho(qx = 2\pi m, t)/\rho_0$, is plotted for this case. From Eq. (29) the largest significant harmonic surviving to the first Talbot time is $j_{\max}^T \sim q/(4k_F) \sim 2 - 3$.

To emphasize the dephasing, Fig. 3 shows a comparison between the density at the first (Fig. 3a), second (Fig. 3b), and fifth (Fig. 3c) focuses ($t \simeq \tau_f$, $\tau_T/2 - \tau_f$, and $\tau_T + \tau_f$, respectively) for $\lambda = 2.5$ and $k_F/q = 0, 0.1, 1.5$. The focal density's peak is diminished and its width broadened by the particle collisions as time progresses. While we calculated $j_{\max}^f \sim 28$ for $k_F/q = 0.1$, j_{\max}^f is approximately 4 for $k_F/q = 1.5$. The first focus for $k_F/q = 1.5$, denoted by the dashed line, is clearly flattened and broadened in Fig. 3a while the focal density for $k_F/q = 0.1$ is indistinguishable from the $k_F/q = 0$ case. By the second focal time in Fig. 3b, the $k_F/q = 1.5$ case has returned to nearly uniform density (i.e., the $t \rightarrow \infty$ limit), and the $k_F/q = 0.1$ case, denoted by the dash-dot line, has started to damp. In Fig. 3c both cases $k_F/q \neq 0$ have nearly uniform densities at $t = \tau_T + \tau_f$, the focus after one Talbot time. From these plots it is clear that a comparison of the normalized density as a function of time can act as a sensitive measure of the time scales for which many-body effects become important compared to single particle effects.

V. PERTURBATION LIMITS AND FUNDAMENTAL PROCESSES

By looking at the perturbative limit of these results, insight is gained into the fundamental processes taking place in this system. In particular, the cases of large spatial period ($q \ll 2k_F$) and small spatial period ($q \gg 2k_F$) pulses can be further distinguished. Taking the perturbative limit of Eq. (29) to first order in λ , only the lowest-order harmonic survives. The result is

$$\rho(x, t) \simeq \rho_0 \left(1 + 4\lambda \sin \left(\frac{\hbar q^2 t}{2M} \right) \cos qx \frac{\sin v_F q t}{\frac{\hbar k_F q t}{M}} \right), \quad (33)$$

where $v_F = \hbar k_F / M$ is the Fermi velocity. This first-order density modulation can be traced to Eq. (20c), taking $e^{\pm i\lambda(\rho_q + \rho_q^\dagger)} \approx 1 \pm i\lambda(\rho_q + \rho_q^\dagger)$. Two time scales are present in this limit, the Talbot time τ_T and the dephasing or collision time $\tau_c \simeq (qv_F)^{-1}$. By examining Eq. (33) for small times, two different pictures of the quantum processes emerge, depending on the size of $q/2k_F$. Note that, for $t/\tau_T \ll 1$ or $t/\tau_c \ll 1$, the signal does not decay.

For $\hbar q^2 t / (2M) \ll 1$, we can rewrite Eq. (33) as

$$\rho(x, t) \simeq \rho_0 \left(1 + 2\lambda \frac{q}{k_F} \cos qx \sin qv_F t \right) \quad (34a)$$

$$= \rho_0 \left(1 + \lambda \frac{q}{k_F} [\sin q[v_F t - x] + \sin q[v_F t + x]] \right). \quad (34b)$$

If this expression is to be valid for times $qv_F t \sim 1$ when $\hbar q^2 t / (2M) \ll 1$, the condition, $q \ll 2k_F$, is necessary. This is the lowest-order quantum scattering limit where interactions dominate. The density modulation has a small amplitude, $2\lambda q / k_F$, and is periodic in time with frequency qv_F . The sinusoidal time dependence can be traced to the uniform Fermi distribution of momenta. Thus, in the limit $q \ll 2k_F$ a small amplitude modulation can be created on a length scale $2\pi/q$ such that $L \gg 2\pi/q \gg d = L/N$. In some sense Eq. (34a) looks like a classical Doppler modulation of the density as a function of time. For $q \ll 2k_F$ the scattering can be seen as an effective process where only the small fraction ($\sim q/k_F$) of particles near the Fermi surface can scatter the pulse due to the exclusion principle, and these particles act independently on each side of the Fermi surface. The pulse creates a superposition of two amplitudes for each of the particles, either with wave vectors $\sim k_F$ and $\sim (k_F + q)$ on the right side of the Fermi surface, or with wave vectors $\sim (-k_F)$ and $\sim (-k_F - q)$ on the left. The density pattern of Eq. (34b) forms as each particle interferes with itself with the same relative phase $\sim qv_F t$ but opposite relative momenta, $\hbar q$ and $-\hbar q$, for the right and left running particles, respectively. At longer times the recoil energy of the scattering, $\hbar^2 q^2 / (2m)$, emerges in Eq. (33), signifying the breakdown of this limit and the decay in time of the modulation amplitude.

In Fig. 4a the density, $\rho(x=0, t)/\rho_0$, is plotted for $\lambda = .05$ and $k_F/q = 6$. For early times the signal is given by Eq. (34a). At later times $\hbar q^2 t / (2M) \sim 1$, a beating with the Talbot oscillations appears as a result of the product of sinusoidal functions in Eq. (33). This beating occurs only because the Fermi distribution is piecewise uniform, giving rise to the time dependence $\sin[qv_F t]/qv_F t$. Careful observation shows that the signal depends only on the beat frequencies, $|2k_F/q \pm 1|/\tau_T = 11/\tau_T, 13/\tau_T$.

For $qv_F t \ll 1$, we can rewrite Eq. (33) as

$$\rho(x, t) \simeq \rho_0 \left(1 + 4\lambda \sin \left(\frac{\hbar q^2 t}{2M} \right) \cos qx \right). \quad (35)$$

If this expression is to be valid for times $\hbar q^2 t / (2M) \sim 1$ when $qv_F t \ll 1$, the condition, $q \gg 2k_F$, is required. This limit recovers the Talbot effect and is the lowest-order quantum scattering limit where single particle effects dominate. Note that the amplitude of the modulation does not depend on the factor k_F . For times $\hbar q^2 t / (2M) \sim 1$ the relative phase from the initial kinetic energy of the fastest particle in the Fermi distribution is insignificant compared to the phase from the recoil energy gained by each particle from the pulse, $|\hbar^2(\pm k_F + q)^2 / (2M) - \hbar^2 k_F^2 / (2M)| \simeq \hbar^2 q^2 / (2M)$. As a result, this looks like a case of quantum scattering from a state having $k \simeq 0$.

In Fig. 4b the density, $\rho(x = 0, t) / \rho_0$, is plotted for $\lambda = .05$ and $k_F / q = .125$. The signal rises at early times as predicted by Eq. (35). At later times $qv_F t \sim 1$, the beat frequencies of Eq. (33), $|2k_F / q \pm 1| / \tau_T = 5 / (4\tau_T), 3 / (4\tau_T)$, are apparent. In Fig. 4c for $\lambda = .05$, we plot the degenerate case, $2k_F = q$, where the two effects oscillate in phase to produce a modulation which goes as $\sin^2(\hbar q^2 t / (2M)) / \hbar q^2 t / (2M)$ at $x = 0$.

VI. CONCLUSION

The exact results showing the focusing and Talbot effects in a one-dimensional condensate may have direct application to a number of neutral atom experiments employing confined Bose-Einstein condensates or beams of cold atoms from a Bose-Einstein condensate [1,6]. Hopefully, some knowledge has been gained into the role of interactions and/or inhomogeneity in causing the decay of coherent properties in these type of systems when probed by transient interactions.

This paper has also shown the power of using traditional condensed matter techniques both to solve new problems in solid state physics and to map problems from atomic or optical physics onto solid state equivalents. The exact density, Eq. (29), is equally valid for a one-dimensional Fermi system at zero temperature and could be adapted for fermions at finite temperatures. Furthermore, the correspondence between fermions and bosons can be extended within this hard core interaction model to include a confining potential in one-dimension, such as a harmonic trap. The many-body ground state wave function of the bosons is the absolute value of a Slater determinant of the first N single-particle energy eigenstates of the confining potential. For the oscillator case the eigenstates are separated by $\hbar\omega$, and the condensate ground state has the total energy $N^2 \hbar\omega / 2$. Whether this ground state corresponds to a true Bose condensate according to the Penrose and Onsager criterion [9] is an open question.

In addition, we have shown that the density formed by the bosons and fermions after the pulse takes the same

form as the density of N independent particles with a uniform, inhomogeneous momentum distribution passing through a standing wave phase grating. As a result, using the analogy with atom optics, we expect that the recoil oscillations and focusing effects in the boson and fermion systems could be recovered with echo techniques by applying a second pulse to rephase the different density harmonics [4].

ACKNOWLEDGMENTS

J.L.C. would like to acknowledge the support of Laura Glick and Professor Max Cohen. This work is supported by the National Science Foundation under Grants No. PHY-9414020 and PHY-9800981, by the U.S. Army Research Office under Grants No. DAAG55-97-0113 and DAAH04-96-0160, and by the University of Michigan Rackham predoctoral fellowship.

APPENDIX A: PROOF OF CORRESPONDENCE BETWEEN BOSON AND FERMION OBSERVABLES

The relationship between free fermions and hard core bosons in one dimension was established by Girardeau [7] and Lenard [8]. Each anti-symmetric fermion energy eigenstate $\phi_e^{(F)}(x_1 \dots x_N)$ is a Slater determinant of N single-particle states. Each many-particle boson energy eigenstate $\phi_e^{(B)}(x_1 \dots x_N)$ has the same energy E_e as its corresponding fermion state. The boson eigenfunctions

$$\phi_e^{(B)}(x_1 \dots x_N) = A(x_1 \dots x_N) \phi_e^{(F)}(x_1 \dots x_N) \quad (\text{A1})$$

are formed by symmetrizing $\phi_e^{(F)}(x_1 \dots x_N)$ with the operator

$$A(x_1 \dots x_N) = \prod_{i,j=1, j>i}^N \text{sign}(x_j - x_i), \quad (\text{A2a})$$

$$A^2 = 1 \quad (\text{A2b})$$

The operator $A(x_1 \dots x_N)$ is a many-particle function taking the values ± 1 and preserving the sign of the boson state under interchange of coordinates.

The following is a proof using these relationships that the (single-particle) density and other operators which are diagonal in coordinate space are the same for certain states of the Bose or Fermi system. The boson density as a function of time is defined as

$$\rho(x, t) = \int \dots \int dx_1 \dots dx_{N-1} \Psi^{*(B)}(x_1 \dots x_{N-1}, x, t) \Psi^{(B)}(x_1 \dots x_{N-1}, x, t). \quad (\text{A3})$$

At zero temperature we assume without loss of generality that the initial boson wave function

$\Psi^{(B)}(x_1 \dots x_{N-1}, x, t = 0)$ is a state prepared from the ground state $\Psi_0^{(B)}(x_1 \dots x_{N-1}, x)$ by some many-particle, unitary operator U ,

$$\Psi^{(B)}(t = 0) = U\Psi_0^{(B)} = UA(x_1 \dots x_{N-1}, x)\Psi_0^{(F)}(x_1 \dots x_{N-1}, x), \quad (\text{A4})$$

where the second equality follows from Eq. (A1) for the fermion ground state $\Psi_0^{(F)}$.

The time-dependent wave function $\Psi^{(B)}(t)$ can be expanded as a superposition of the boson energy eigenstates,

$$\Psi^{(B)}(x_1 \dots x_{N-1}, x, t) = \sum_e C_e e^{-iE_e t/\hbar} \phi_e^{(B)}(x_1 \dots x_{N-1}, x), \quad (\text{A5})$$

where the time-independent expansion coefficients

$$C_e = \int \dots \int dx_1 \dots dx_{N-1} dx \phi_e^{*(B)}(x_1 \dots x_{N-1}, x) \Psi^{(B)}(x_1 \dots x_{N-1}, x, t = 0) \quad (\text{A6})$$

are defined by the boson wave function at $t = 0$. When we insert Eq. (A5) into Eq. (A3), the density takes the form

$$\rho(x, t) = \sum_{e, e'} \int \dots \int dx_1 \dots dx_{N-1} C_{e'}^* C_e e^{-i(E_e - E_{e'})t/\hbar} \phi_{e'}^{*(B)}(x_1 \dots x_{N-1}, x) \phi_e^{(B)}(x_1 \dots x_{N-1}, x). \quad (\text{A7})$$

Using Eqs. Eq. (A1), (A2) and (A4), we can rewrite Eq. (A7) as

$$\rho(x, t) = \sum_{e, e'} \int \dots \int dx_1 \dots dx_{N-1} C_{e'}^* C_e e^{-i(E_e - E_{e'})t/\hbar} \phi_{e'}^{*(F)}(x_1 \dots x_{N-1}, x) \phi_e^{(F)}(x_1 \dots x_{N-1}, x), \quad (\text{A8})$$

and the expansion coefficients of Eq. (A6) as

$$C_e = \int \dots \int dx_1 \dots dx_{N-1} dx \phi_e^{*(F)} AU A \Psi_0^{(F)}. \quad (\text{A9})$$

The key point of the proof arises here. In order to write the density of Eq. (A8) entirely in terms of the Fermi system, Eq. (A9) can not contain the symmetrization function A . This implies that A must commute with U ,

$$[U, A(x_1 \dots x_{N-1}, x)] = 0, \quad (\text{A10})$$

giving the coefficients

$$C_e = \int \dots \int dx_1 \dots dx_{N-1} dx \phi_e^{*(F)} U \Psi_0^{(F)}. \quad (\text{A11})$$

Condition (A10) is satisfied if U depends only on the coordinates $x_1 \dots x_{N-1}, x$. (In particular, the pulse operator from Eq. (13) above,

$$U = \exp[i2\lambda \sum_j \cos(qx_j)/\hbar], \quad (\text{A12})$$

obeys condition (A10).) If this is the case, the time-dependent density is defined by Eqs. (A8) and (A11) for both Bose and Fermi systems prepared by the operator U .

In general, when Eq. (A11) is inserted into (A8), the density can be written in second quantized form as

$$\langle F | U^\dagger e^{iH_0 t/\hbar} c_x^\dagger c_x e^{-iH_0 t/\hbar} U | F \rangle. \quad (\text{A13})$$

This is Eq. (20a) from the text. While the density operator in second quantized form $c_x^\dagger c_x$ is automatically diagonal, from Eq. (A13) it is clear that if condition (A10) is obeyed, the expectation value of any operator which is diagonal in x can be calculated using this boson-fermion correspondence.

-
- [1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, *Science* **269**, 198 (1995); C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, *Phys. Rev. Lett.* **75**, 1687 (1995); K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, *Phys. Rev. Lett.* **75**, 3969 (1995); D.G. Fried, T.C. Killian, L. Willmann, D. Landhuis, S.C. Moss, D. Kleppner, and T.J. Greytak, *Phys. Rev. Lett.* **81**, 3807 (1998); P. Zoller, *Phys. Rev. Lett.* **81**, 3807 (1998)
 - [2] M. Lewenstein and L. You, *Phys. Rev. Lett.* **71**, 1339 (1993); M. Edwards and K. Burnett, *Phys. Rev. A* **51**, 1382 (1995); J. Javanainen, *Phys. Rev. Lett.* **72**, 1927 (1995); K.G. Singh and D.S. Rokhsar, *Phys. Rev. Lett.* **77**, 1667 (1996); M. Edwards, P.A. Ruprecht, K. Burnett, R.J. Dodd, and C.W. Clark, *Phys. Rev. Lett.* **77**, 1671 (1996); J. Javanainen, J. Ruostekoski, B. Vestergaard, and M.R. Francis, *Phys. Rev. A* **59**, 649 (1998)
 - [3] P.E. Moskowitz, P.L. Gould, S.R. Atlas, and D.E. Pritchard, *Phys. Rev. Lett.* **51**, 370 (1983); B. Dubetsky, V.P. Chebotayev, A.P. Kazantsev, and V.P. Yakovlev, *JETP Lett.* **39**, 649 (1984); D.W. Keith, C.R. Ekstrom, Q.A. Turchette, and D.E. Pritchard, *Phys. Rev. Lett.* **66**, 2693 (1991); E.M. Rasel, K. Oberthaler, H. Batelaan, J. Schmiedmayer, and A. Zeilinger, *Phys. Rev. Lett.* **75**, 2633 (1995); D.M. Giltner, R.W. McGowan, and S.A. Lee, *Phys. Rev. Lett.* **75**, 2638 (1995); M.S. Chapman, C.R. Ekstrom, T.D. Hammond, J. Schmiedmayer, B.E. Tannian, S. Wehinger, and D.E. Pritchard, *Phys. Rev. A* **51**, R14 (1995)
 - [4] S. Cahn, A. Kumarakrishnan, U. Shim, T. Sleator, P. R. Berman and B. Dubetsky, *Phys. Rev. Lett.* **79**, 784 (1997)
 - [5] For a review of atom optics stressing laser-based optical elements, see C. Kurtsiefer, R.J.C. Spreeuw, M. Drewsen, M. Wilkens, and J. Mlynek in *Atom Interferometry*, ed. by P.R. Berman, Academic Press, San Diego (1997)

- [6] M.-O. Mewes, M.R. Andrews, D.M. Kurn, D.S. Durfee, C.G. Townsend, and W. Ketterle, Physical Review Letters **78**, 582 (1997); M. Kozuma, L. Deng, E.W. Hagley, J. Wen, K. Helmerson, S.L. Rolston, and W.D. Phillips (preprint)
- [7] M. Girardeau, J. Math. Phys, **1**, 516 (1960).
- [8] A. Lenard, J. Math. Phys. **5**, 930 (1964).
- [9] R. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).
- [10] K. Patorski, Progress in Optics XXVII, 1 (1989)
- [11] H.F. Talbot, Philos. Mag. **9**, 401 (1836)
- [12] U. Janicke and M. Wilkens, J. Phys II (France) **4**, 1975 (1994)
- [13] G. Timp, R. E. Behringer, D. M. Tennant, J. E. Cunningham, M. Prentiss, K. Berggren, Phys. Rev. Lett. **69**, 1636 (1992); T. Sleator, V. Balykin, and J. Mlynek, Appl. Phys. B **54**, 375 (1992); J.J. McClelland, R.E. Scholten, E.C. Palm, and R.J. Celotta, Science **262**, 877 (1993)
- [14] J.L. Cohen, B. Dubetsky, and P.R. Berman (to be published)

FIG. 1. Density Plots, $\rho(x,t)/\rho_0$, for $\lambda = 2.5$, $-0.5 \leq qx/2\pi \leq 0.5$ (horizontal axis), and $0 \leq t/\tau_T \leq 0.5$ (vertical axis). (a) $k_F/q \rightarrow 0$, (b) $k_F/q = 0.1$. The Talbot effect is evident in (a) but washed out in (b). (Grey scale: white is high density and black is low density.)

FIG. 2. Density as a function of time from Fig. 1 along the planes $qx = 2\pi m$: $\rho(0, t)/\rho_0$. (a) $k_F/q \rightarrow 0$, (b) $k_F/q = 0.1$. In (b) the focusing effect is clearly suppressed at later times as higher harmonics damp out more quickly than lower harmonics. Harmonics up to $j_{\max}^T \sim 2-3$ have significant amplitudes near $t = \tau_T$.

FIG. 3. Focal density for $\lambda = 2.5$ over one period, $-0.5 \leq qx/2\pi \leq 0.5$, for Fermi wave vectors, $k_F/q = 0$ (solid), 0.1 ($-\cdot-\cdot-$), and 1.5 ($- - -$), at times (a) $t \simeq \tau_f$, (b) $\tau_T/2 - \tau_f$, and (c) $\tau_T + \tau_f$. At the first focus (a) the $k_F/q = 0.1$ density is indistinguishable from that at $k_F/q = 0$ while the $k_F/q = 1.5$ case is somewhat damped. By the second focal time (b) the $k_F/q = 0.1$ density has begun to damp while the $k_F/q = 1.5$ density shows little spatial modulation. After one Talbot time the focuses in (c) for both cases with interactions are severely damped.

FIG. 4. Perturbation Theory. The density along the planes $qx = 2\pi m$ from the exact result [Eq. (29)], $\rho(0, t)/\rho_0$, is plotted versus time for $\lambda = 0.05$. The lowest-order spatial harmonic dominates the modulated density in all cases, reproducing the perturbative result [Eq. (33)]. The beating between the Talbot period τ_T and the dephasing period $\tau_T/(2k_F/q)$ is evident. (a) $k_F/q = 6$ - the density goes like Eq. (34a) for times $t/\tau_T \ll 1$ but $(2k_F/q)(t/\tau_T) \sim 1$. (b) $k_F/q = 0.125$ - Talbot oscillations dominate according to Eq. (35) for times $(2k_F/q)(t/\tau_T) \ll 1$ but $t/\tau_T \sim 1$. (c) Degenerate case, $k_F/q = 0.5$ - the Talbot and dephasing oscillation are in phase with the single time scale τ_T . In all cases the signal begins to decay as the dephasing oscillations beat against the Talbot oscillations.